SENSITIVITY ANALYSIS FOR THE OPTIMAL DESIGN AND CONTROL OF ADVANCED GUIDANCE SYSTEMS

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The main goal of the research project is to use Continuous Sensitivity Equation Methods in order to design actuators and sensors for distributed parameter systems. Investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types mathematical models. When the governing equations are partial differential equations and the sensitivity analysis is with respect to parameters that determine placement of sensors and/or actuators, there can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. This issue is particularly important for accuracy and convergence of numerical sensitivity calculations that may be used within a control design framework.

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Abstract

The main goal of the research project is to use sensitivity analysis to design actuators and sensors for distributed parameter systems (DPS). The choice of sensor/actuator for a control problem, its action and placement thereof, is addressed by computing the sensitivity of the state variables with respect to small changes in a parameter characterizing placement. Investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types of models. Parameters determining placement tend to enter the PDE models in ways that affect the differentiability of the state variables. There can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. When using Continuous Sensitivity Equation Methods (CSEMs), the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Simple models involving actuator placement contain discontinuous coefficients, and the explicit parameter location dependence of these discontinuities provides an *a priori* indication that sensitivity variables lack the same smoothness properties as those of the state. The issue is particularly important for accuracy and convergence of numerical sensitivity calculations.

Contents

1   Technical Summary

This work concerns the development of computational tools to incorporate into the design of control systems—control law, observers, choice and placement of sensors and actuators—for flexible wing MAVs. Such vehicles have been the focus of research efforts supported by AFOSR and have unique capabilities that can be utilized in urban missions. Capabilities such as flight control via wing shaping have been demonstrated, and it is known that these MAVs can fly at increased angle of attack without stall. However, fundamental aerodynamics of such vehicles is not well-understood, and computational studies to enhance the of control systems are now being developed. A vital contribution to that effort is the choice and placement of sensors and actuators used within the control regime.

The main goal of the research project is to use sensitivity analysis to design actuators and sensors for distributed parameter systems (DPS). The choice of sensor/actuator for a control problem, its action and placement thereof, is addressed by computing the sensitivity of the state variables with respect to small changes in a parameter characterizing placement. Preliminary investigations for parameterized sensor/actuator placement indicate that computational challenges exist for certain types of models. Parameters determining placement tend to enter the PDE models in ways that affect the differentiability of the state variables. There can be a loss of regularity between the model equations and that of the corresponding sensitivity equations. When using Continuous Sensitivity Equation Methods (CSEMs), the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Simple models involving actuator placement contain discontinuous coefficients, and the explicit parameter location dependence of these discontinuities provides an *a priori* indication that sensitivity variables lack the same smoothness properties as those of the state variables. The issue is particularly important for accuracy and convergence of numerical sensitivity calculations.

For the last three years, my collaborators and I have investigated various aspects of the use of sensitivity analysis for locating sensors and actuators. Computational models are required for control design and sensitivity calculations (for sensor and actuator selection and placement). The focus of this research has been the mathematical analysis and development of computational tools for sensitivity approximation for models where the parameter of interest determines sensor/actuator placement. Investigations indicate that computational challenges exist for certain types of models, and one important indicator of such challenges is the manner in which the parameters appear in the underlying mathematical models. Parameters determining placement tend to enter the pde models in ways that affect the differentiability of the state variables with respect to small changes in such parameters. At the very least, there may be a loss of regularity between the model equations and that of the corresponding sensitivity equations. In Section 1.6, we discuss a proof-of-concept example which shows how sensitivity computations can be obtained efficiently and can be validated if the scale of the problem allows. In Section 1.7, an Euler-Bernoulli Beam model with patch actuators is presented in order to illustrate the computational challenges that may occur, and a corresponding steady state model problem is also examined to identify some of the mathematical issues.

1.1   Methodology and Theory Used in the Research

In this section, an overview of the types of models that have been examined in the research is given. Notation, methodology and theory is also outlined and will be referenced throughout

the report.

1.2 Partial Differential Equation Models and Their Discretizations

When we speak of a "model", we are referring to a state equation that describes the governing physics of the problem to be controlled. Since the focus of this report examines actuator placement, we omit the measurement equation that models the sensor dynamics. Specifically, we consider the simplified underlying model considered in this research to be a linear system of partial differential equations of the form

$$\dot{x}(t) \;\; = \;\; Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1.1}$$

where $x(t)$ is the state in the state space X and $u(t)$ is the control effected by the actuators. The operator A represents the linear dynamics of the system, and B represents the physics of the actuator input to the system. Depending on the application, the state vector may be composed solely of flow velocities, or perhaps velocities and structural components if we are considering the coupling between the vehicle and flow. In a more general discussion, one would include several important terms in the model, most importantly, a nonlinear operator describing the nonlinear dynamics of the system to be modelled. Other terms such as a disturbance term that can be used to incorporate noise or unmodeled dynamics as well as a measurement term that provides a measurement of the state provided by the sensors should also appear. However, for the discussion given in this report we simplify the governing equation to a linear model and focus on the most interesting aspects of the sensitivity analysis research that has been conducted for this project.

This type of model will be used as guidance in algorithm development. For example, theory for sensitivity analysis for distributed parameter systems uses the PDE model as the starting point [17, 69, 71, 72, 74]. Similarly, design of convergent control designs relies on properties of the underlying PDE system.

Deriving a finite dimensional model, one can obtain a discretized version of (1.1) of the form

$$\dot{x}^N(t) \;\; = \;\; A^N x^N(t) + B^N u^N(t), \quad x^N(0) = x_0^N \tag{1.2}$$

where N is related to the grid used in discretization, and the state vector $x^N(t)$ is obtained by applying an appropriate approximation scheme such as a finite element discretization.

1.3 System Sensitivities

In this research project, we make extensive use of the sensitivity of the state of the system, or control, to a parameter of interest, α. The parameter may represent a sensor or actuator location, its length, or some other characteristic that would affect the overall closed loop system performance and or robustness. If we denote the sensitivity of the state variable

$$x_\alpha(t) = \frac{\partial x(t)}{\partial \alpha}$$

then we can derive the sensitivity equation for the system defined in (1.1) by implicitly differentiating to yield

$$\dot{x}_\alpha(t) = Ax_\alpha(t) + A_\alpha x(t) + Bu_\alpha(t) + B_\alpha u(t), \tag{1.3}$$

$$x_\alpha(0) = 0.$$

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Issues with respect to existence, regularity, and calculation of the sensitivities for particular systems provide fundamental questions for research. Controller sensitivities will be presented after the overview of the basic control method given in the following section.

1.3.1   The Linear Quadratic Regulator Problem

Given a system defined by any of the models described by equations (1.1) or by (1.2), the associated linear quadratic regulator (LQR) problem is as follows:
Determine the control $u(t)$ in an admissible set, U, that minimizes the cost functional

$$J(u) = \int_0^\infty \langle Qx, x \rangle_X + \langle Ru, u \rangle_U \, dt,$$

subject to the constraint that $x(t)$ in state space X satisfies the state equation, and Q and R are state and control weighting operators, respectively. Assuming full state feedback, optimal control theory gives the feedback control in the form

$$u(t) = -R^{-1}B^*\Pi x(t) = -Kx(t), \tag{1.4}$$

where K is the gain operator, and Π is the solution of the control algebraic Riccati equation

$$A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0. \tag{1.5}$$

Functional Gains
    For many PDE control problems, the control law (1.4) can be written as

$$u(t) = -[Kx(t)] = \int_\Omega k(s)x(t,s)ds, \tag{1.6}$$

for example see [19, 20, 48, 50]. In the case that there is more than one actuator input, e.g., more at more than one spatial location, ξ, there will be a functional gain for each location and the inputs can be represented as

$$u_\xi(t) = -[Kx(t)](\xi) = \int_\Omega k_\xi(s)w_c(t,s)ds. \tag{1.7}$$

The kernels $k_\xi(s)$ are called functional gains, and have been utilized for reduced order controller design and sensor placement in [4, 5, 21, 41, 49, 53].

1.4   Control System Sensitivities

Computing sensitivities of the control law with respect to parameters is one of the primary research interests. If we define the control sensitivity

$$u_\alpha(t) = \frac{\partial u}{\partial \alpha}$$

then the associated sensitivity system is given by equation (1.3) and

$$u_\alpha(t) \;\; = \;\; -R^{-1}\left[B^*\Pi x_\alpha(t) + (B^*\Pi_\alpha + B_\alpha^*\Pi)x(t)\right],$$

where Π_α solves the Lyapunov equation

$$\hat{A}^*\Pi_\alpha + \Pi_\alpha \hat{A} = \hat{Q}, \tag{1.8}$$

with

$$\hat{A} = A - BR^{-1}B^*\Pi \ \text{ and } \ \hat{Q} = -A_\alpha^*\Pi - \Pi A_\alpha + \Pi(BR^{-1}B_\alpha^* + B_\alpha R^{-1}B^*)\Pi - Q_\alpha.$$

This Lyapunov equation is derived by implicitly differentiating through the Riccati equation (1.5) with respect to α. Note the use of the notation $A_\alpha = \frac{\partial A}{\partial \alpha}$ for the sensitivity of the A operator, and similarly for B_α and Q_α. We note that in general, the control weighting, R, does not depend on α.

1.5 Results of Funding

1.6 Investigations on Sensitivity Computations for the Heat Equation with Respect to Actuator Placement

This section addresses an actuator design problem for the simple 1D heat equation. The idea is that the choice and placement of sensors and actuators for a control problem could be addressed by computing the sensitivity of the state of the system with respect to a characteristic of the actuator placement. To illustrate the ideas, we consider the sensitivity of the state variable, temperature in this case, to actuator location. This work provides the preliminary computations that demonstrate proof-of-concept for the combination of CSEMs with basic optimal control techniques. The sensitivity of the temperature with respect to actuator placement is investigated using a Continuous Sensitivity Equation Method, see [17, 43, 71, 72, 74, 75] for examples. Consider the heat equation model for the situation when an actuator location is specified by the parameter α:

$$w_t(t,\xi) = \epsilon w_{\xi\xi}(t,\xi) + b(\xi;\alpha)u(t) \tag{1.9}$$

$$w(t,0) = 0, \ w(t,1) = 0$$

$$w(0,\xi) = w_0(\xi) \tag{1.10}$$

with

$$b(\xi;\alpha) = e^{-(\xi-\alpha)^2}.$$

Defining the state variable $x(t) = w(t,\cdot)$ and using standard PDE theory, one can formulate (1.9)-(1.10) in the form of (1.1) posed in an appropriate Sobolev space. We forego the presentation of these details for the sake of brevity. However, we note that the B operator above depends explicitly on the parameter of interest α; this fact is important for the sensitivity analysis that follows.

Many of the computational issues related to sensitivity analysis depend heavily on the correct mathematical formation for the problem. Here we outline the application of the Continuous Sensitivity Equation Method (CSEM) to the LQR control problem for the heat equation given above. The following discussion uses the temperature (state) sensitivity, the Π operator sensitivity and the controller sensitivity, respectively, defined as

$$w_\alpha(t,\xi) = \frac{\partial w(t,\xi)}{\partial \alpha}, \quad \Pi_\alpha = \frac{\partial \Pi}{\partial \alpha}, \quad u_\alpha(t) = \frac{\partial u(t)}{\partial \alpha}.$$

To derive the appropriate sensitivity equations, one implicitly differentiates equations (1.9)-(1.10) with respect to the parameter α. The resulting sensitivity equation with boundary and initial conditions is given by

$$(w_\alpha)_t(t,\xi) = \epsilon(w_\alpha)_{\xi\xi}(t,\xi) + 2(\xi-\alpha)b(\xi;\alpha)u(t) + b(\xi;\alpha)u_\alpha(t) \tag{1.11}$$

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$$w_\alpha(t,0) = 0, \ w_\alpha(t,1) = 0$$

$$w_\alpha(0,\xi) = 0.$$

Further, we note that this equation can be posed in a simplified form of equation (1.3) as

$$\dot{x}_\alpha(t) = Ax_\alpha(t) + Bu_\alpha(t) + B_\alpha u(t), \quad x_\alpha(0) = 0 \tag{1.12}$$

The state feedback control system is given by equations (1.1), and the optimal feedback control by (1.4) determined by the algebraic Riccati equation (1.5). The control sensitivity can be computed as described in Section 1.4. In this example, both operators, B and B_α, have integral representations, and it is an easy calculation to see that the B operator varies in a differentiable manner with respect to the actuator location parameter α. In the following section we outline the discretization and sensitivity approximations for both the state variable as well as the control.

1.6.1 Numerical Experiments

Here, a brief summary of the numerical experiments for the actuator placement modelis given. The initial condition used for the numerical calculations is the piecewise constant function

$$w_0(\xi) = w(0,\xi) = \begin{cases} -1, & \xi \in [0,0.5] \\ 1, & \xi \in (0.5,1.0], \end{cases}$$

and the thermal diffusivity coefficient for all of the numerical results presented here is given by $\epsilon = 0.01$. The state equation in (1.12) is discretized using a standard finite element method in the spatial coordinate with cubic B-spline basis functions, and a time solver from Matlab's suite of ODE solvers is used for the time stepping. As this is a fairly standard treatment of the discretization and simulation, we omit details. Matlab's `lqr` and `lyap` functions are used to solve the algebraic Riccati equation to approximate Π and to approximate the Riccati sensitivity, Π_a, respectively.

In Figure 1, an example of a temperature approximation using piecewise cubic basis functions is shown for the case where $\alpha = 0.7$ and $N = 40$ basis functions are used. Figure 2 contains the corresponding temperature sensitivity for those same parameter values. Note that the sensitivity gives the qualitative information that one would expect for the case when $\alpha = 0.7$. That is, the behavior of w_α in Figure 2 shows that the temperature w is most sensitive in the region around $\xi = 0.7$, and that the temperature is most sensitive for early values of t before the diffusion and the controller have driven the temperature to be very near equilibrium. The associated control and control sensitivity computations are given in Figure 3 and 4. The behavior of these functions is also as expected. The control effort is large initially and quickly decays to 0. Similarly, the control is most sensitive to small changes in α for small values of t. As the t-variable increases, the control is less sensitive to changes in α. In the following section, we include a brief discussion on the effort to validate the sensitivity calculations for this simple example.

1.6.2 Validation of Sensitivity Calculations

This section briefly demonstrates that for the simple example of the 1D heat equation, one can validate the sensitivity calculations shown in the previous section. The following shows that this can be done for the temperature, the control variable, or the Riccati operator approximations. To clarify, the numerical approximations of the sensitivities for
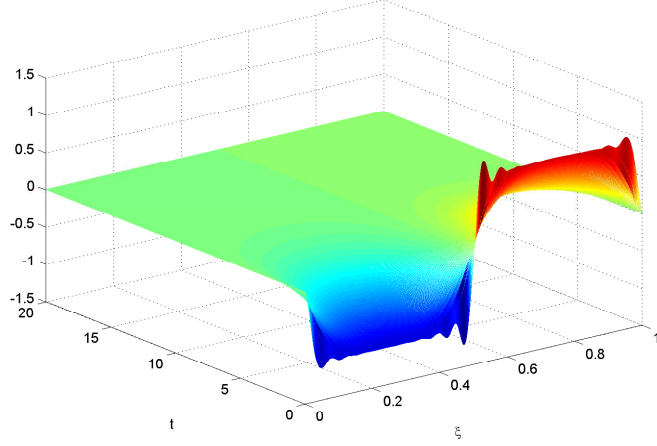
Figure 1: Temperature Approximation $w(t, \xi)$ with N = 40 and $\alpha = 0.7$

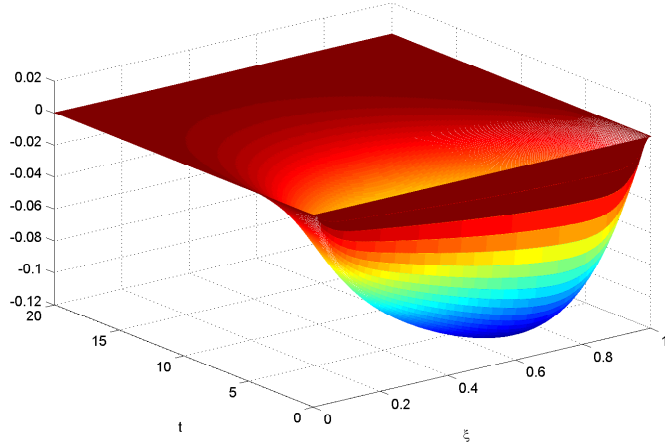
Figure 2: Sensitivity Approximation $w_\alpha(t, \xi)$ with N = 40 and $\alpha = 0.7$

various variables are compared with finite difference calculations. For example, the control sensitivity, u_α is compared with a centered finite difference approximation so that

$$u_\alpha^N(t; \alpha) \approx \frac{u^N(t; \alpha + \Delta\alpha) - u^N(t; \alpha - \Delta\alpha)}{2\Delta\alpha}.$$

The superscript notation u^N refers to the fact that the calculation is based on a finite element discretization in space, and for the results presented below, a reasonably fine mesh of $N = 40$ is used in order to guarantee an accurate approximation for the variables of interest. Hence, the convergence that we present here refers to a convergence of the finite difference approximation as one allows $\Delta\alpha \to 0$. Table 1 gives results for each of the sensitivity variables w_α, u_α, and Π_α for a specific value of the α parameter. Similar results were observed for a range of parameter values. Although sensitivity calculations obtained via
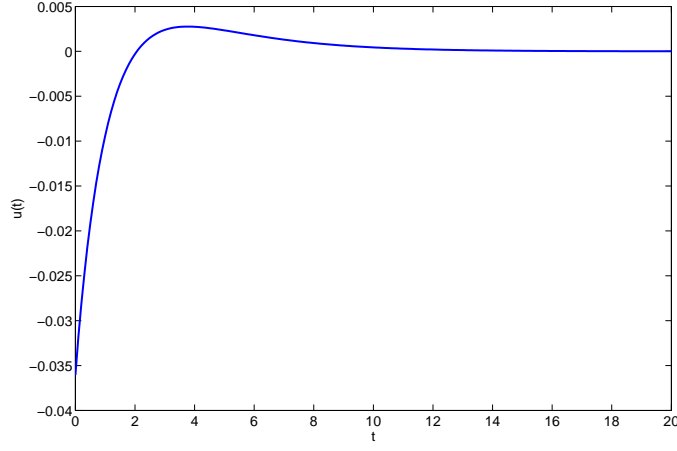
Figure 3: Control Approximation $u(t)$ with N = 40 and $\alpha = 0.7$
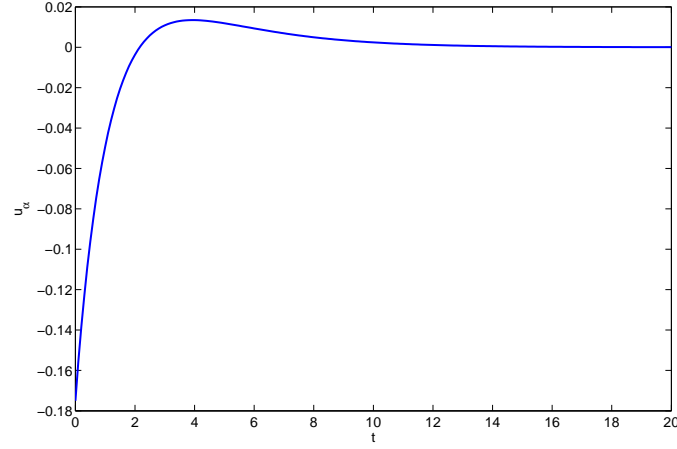
Figure 4: Control Sensitivity Approximation $u_\alpha(t)$ with N = 40 and $\alpha = 0.7$

finite difference techniques can be difficult to obtain for large scale problems, it is appropriate to validate our sensitivity calculations whenever possible. For the case of this simple model problem, sensitivity calculations obtained by the CSEM approach are validated by comparison to centered finite difference sensitivity computations. In the following section, one will see why such a validation is not always feasible for certain types of sensitivity variables, particularly for certain types of sensitivity variables related to actuator location for some mathematical models. An investigation of actuator location for a boundary control problem with the 2D heat equation can by found in [32].

Step Size	State ($\alpha = 0.7$)	Control ($\alpha = 0.7$)	Riccati ($\alpha = 0.2$)
$\Delta\alpha = 10^{-1}$	2.0600×10^{-2}	3.1782×10^{-4}	3.1000×10^{-3}
$\Delta\alpha = 10^{-2}$	2.0782×10^{-5}	3.1802×10^{-7}	3.0808×10^{-5}
$\Delta\alpha = 10^{-3}$	1.0914×10^{-6}	3.1801×10^{-10}	3.0808×10^{-7}
$\Delta\alpha = 10^{-4}$	1.0900×10^{-7}	3.1418×10^{-13}	2.9993×10^{-9}
$\Delta\alpha = 10^{-5}$	1.0900×10^{-8}	1.4816×10^{-14}	

Table 1: Error Calculations Showing Convergence of Finite Difference Calculations for w_α, u_α and Π_α.

1.7 Computational Issues with Sensitivity Calculations for Actuator Placement on an Euler-Bernoulli Beam

In this section we discuss an actuator placement problem for a cantilevered Euler-Bernoulli beam with piezoceramic patch actuators as in [8, 9, 10]. We consider the sensitivity of the state—position and velocity of the beam—to patch location and to patch length. This work provides the preliminary computations which would then be used for optimal selection of patch length and location.

The framework that is currently being developed combines CSEMs, optimization and LQR control techniques to gain insight into sensor and actuator placement for control design for physical systems governed by a partial differential equations. However, this section shows that the sensitivity calculations for actuator placement can present many mathematically interesting and numerically challenging problems.

This discussion relates preliminary sensitivity calculations for a cantilevered Euler-Bernoulli beam model with a pair of piezoceramic patch actuators placed on either side of the beam at the same specified spatial location (see Figure 5). We consider the case when the patches are excited out-of-phase which results in pure bending of the beam. As in the previous example, the sensitivity of the displacement of the beam with respect to patch placement is investigated using the CSEM. We incorporate both Kelvin-Voigt and
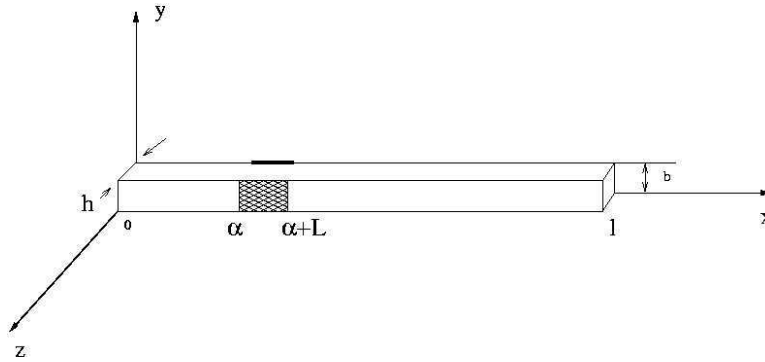
Figure 5: Diagram of beam with a pair of piezoceramic patches of length L located on either side of the beam on the interval $x \in [\alpha, \alpha + L]$.

linear viscous (air) damping. The length of the beam is ℓ, the height is given by b and the thickness is denoted as h. The length of the patch is given by L, and the patch location is given by the interval $[\alpha, \alpha + L]$. Let $w(t, x)$ denote the displacement of the beam at

time t and position x. The motion of the cantilevered beam is governed by the PDE with boundary conditions

$$\rho A w_{tt} + d_{lv} w_t + (d_{kv} I w_{txx})_{xx} + (EI w_{xx})_{xx} = g(t, x), \qquad (1.13)$$

$$w(t, 0) = w_\xi(t, 0) = 0 \qquad (1.14)$$

$$EI w_{xx}(t, \ell) + d_{kv} I w_{txx}(t, \ell) = 0, \qquad (1.15)$$

$$(EI w_{xx})_x(t, \ell) + (d_{kv} I w_{txx})_x(t, \ell) = 0. \qquad (1.16)$$

and with the initial deflection and initial velocity denoted by

$$w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x). \qquad (1.17)$$

The coefficients all represent material properties of the beam at a certain spatial location x: ρ is the mass density, A is the cross-sectional area, I is the moment of inertia, E is Young's modulus, and d_{lv} and d_{kv} are the coefficients of air and Kelvin-Voigt damping, respectively.

The presence of the patches results in discontinuities in these coefficients, which can be expressed as follows

$$\rho A(x) = \rho A_1 + \rho A_2 [H_1(x) - H_2(x)], \qquad (1.18)$$

$$d_{kv} I(x) = d_{kv} I_1 + d_{kv} I_2 [H_1(x) - H_2(x)], \qquad (1.19)$$

$$EI(x) = EI_1 + EI_2 [H_1(x) - H_2(x)]. \qquad (1.20)$$

The notation $H_1(x)$ denotes the Heaviside function taking on the value of 0 for $x \in [0, \alpha]$ with jump discontinuity located at $x = \alpha$, the left end of the patch, and H_2 denotes the heaviside function with jump discontinuity located at $x = \alpha + L$, the right end of the patch. The constants $\rho A_1, EI_1$, and $d_{kv} I_1$ correspond to the density, flexural rigidity, and Kelvin-Voigt damping properties of the beam, while the constants $\rho A_2, EI_2$, and $d_{kv} I_2$ correspond to those of the patch, respectively. The patches influence the beam by exerting a moment force on that section. Therefore the spatial influence of the control is described by a difference of Heaviside functions, and the control function for the beam is given by

$$g(t, x) = \kappa [H_1(x) - H_2(x)]_{xx} u(t). \qquad (1.21)$$

The constant κ is a parameter characterizing the patch properties, and $u(t)$ is the voltage applied to the patch at time t. For a more thorough treatment of the model development and the specific forms of the beam and patch parameters, see [10].

One should observe that in the beam equation (1.13), the damping $(d_{kv} I w_{txx})_{xx}$, stiffness $(EI w_{xx})_{xx}$, and control term $g(t, x)$ all contain spatial derivatives of the Heaviside functions; moreover, the actuator location parameter is embedded in the definition of these Heaviside functions. Consequently, the PDE should be mathematically interpreted using the variational (or weak) formulation. The weak form is also convenient for the numerical simulations which use the finite element method for approximation in space. The variational formulation of the beam equation is given in [10]

$$\int_0^\ell \rho A(x; \alpha) w_{tt}(t, x) \phi_{xx}(x) \, dx + \int_0^\ell d_{kv} I(x; \alpha) w_{txx}(t, x) \phi_{xx}(x) \, dx$$

$$+ \int_0^\ell EI(x; \alpha) w_{xx}(t, x) \phi_{xx}(x) \, dx = \int_0^\ell \tilde{g}(t, x; \alpha) \phi_{xx}(x) \, dx, \qquad (1.22)$$

where
$$\tilde{g}(t, x; \alpha) = \kappa[H_1(x) - H_2(x)]u(t).$$

We use this formulation below to discuss computational difficulties with numerically approximating the sensitivities.

1.7.1   Computational Difficulties with the Sensitivity Equation

The parameter of interest for this study is the parameter determining the patch location, α, and we consider both the displacement and the control term to depend explicitly on α. We denote this dependence as $w(t, x) = w(t, x; \alpha)$. The sensitivity of the state with respect to the patch location is defined by

$$w_\alpha(t, x; \alpha) = \frac{\partial w}{\partial \alpha} w(t, x; \alpha).$$

In [73], the PI numerically approximated w_α by deriving a variational sensitivity equation. This equation was obtained by differentiating through the above variational equation (1.22) for the state with respect to α. This procedure causes derivatives of the Heaviside and delta functions to appear in the sensitivity equation. The regularity of the sensitivity equation can play an important role in determining the appropriate computational treatment of the problem.

   To illustrate the main difficulties with the beam sensitivity equation, we remove the time dependence to focus on the issue of regularity in the spatial domain. We also remove one spatial discontinuity from the piecewise constant functions to further simplify the presentation. These modifications allow us to study a similar problem which has a closed form solution for the state and sensitivity. Therefore, instead of the variational equation (1.22), consider the following steady problem: find $w \in V = \{v \in H^2(0, \ell) \colon v(0) = 0, v_x(0) = 0\}$ such that

$$\int_0^\ell EI(x; \alpha) w_{xx}(x) \phi_{xx}(x)\, dx = \int_0^\ell \tilde{g}(x; \alpha) \phi_{xx}(x)\, dx \tag{1.23}$$

for all $\phi \in V$. The functions $EI(x; \alpha)$ and $\tilde{g}(x; \alpha)$ are piecewise constant and are given by

$$EI(x; \alpha) = \begin{cases} EI_1, & 0 < x < \alpha \\ EI_2, & \alpha < x < \ell \end{cases}, \quad \tilde{g}(x; a) = \begin{cases} g_1, & 0 < x < \alpha \\ g_2, & \alpha < x < \ell \end{cases}.$$

This problem can be posed in an abstract differential form; however, like the time dependent beam problem, it is best understood in weak form due to the discontinuities in the coefficient functions EI and $\tilde{g}$.

   We assume $EI_1, EI_2 > 0$ which implies the problem has a unique solution by the Lax-Milgram Theorem. It can be checked that the solution is given by

$$w(x; \alpha) = \begin{cases} b_1 x^2/2, & 0 < x < \alpha \\ b_2 x^2/2 + \alpha(b_1 - b_2)x - \alpha^2(b_1 - b_2)/2, & \alpha < x < \ell \end{cases}, \tag{1.24}$$

where $b_i = g_i/EI_i$ for $i = 1, 2$. The solution is differentiable with respect to α and the sensitivity $w_\alpha(x; \alpha)$ is given by

$$w_\alpha(x; \alpha) = \begin{cases} 0, & 0 < x < \alpha \\ (b_1 - b_2)x - \alpha(b_1 - b_2), & \alpha < x < \ell \end{cases}. \tag{1.25}$$

The sensitivity is continuous and differentiable in x with derivative

$$(w_\alpha)_x(x;a) = \begin{cases} 0, & 0 < x < \alpha \\ b_1 - b_2, & \alpha < x < \ell \end{cases}.$$

If $b_1 \neq b_2$, then $(w_\alpha)_x$ is not differentiable and therefore $w_\alpha \notin V$. Therefore, the sensitivity does not have the same spatial regularity as the state.

This observation has implications for computing the sensitivities. We can proceed in the same manner as described above for the beam problem and formally differentiate the steady problem (1.23) with respect to α to obtain the sensitivity equation

$$\int_0^\ell EI(x;\alpha)(w_\alpha)_{xx}(x;\alpha)\phi_{xx}(x)\,dx + (EI_1 - EI_2)w_{xx}(\alpha)\phi_{xx}(\alpha) = (g_1 - g_2)\phi_{xx}(\alpha). \quad (1.26)$$

Recall that ϕ is in $H^2(0,\ell)$ and so the pointwise evaluations $\phi_{xx}(\alpha)$ do not make sense in general. Also, a direct computation shows

$$w_{xx}(x) = \begin{cases} b_1/2, & 0 < x < \alpha \\ b_2/2, & \alpha < x < \ell \end{cases}$$

so that the term $w_{xx}(\alpha)$ appearing in the sensitivity equation does not have meaning when $b_1 \neq b_2$. In particular, the function w possesses a second derivative only in the weak sense and not in the classical sense of differentiability.

Despite these observations, we can follow the approach taken in [73] and numerically approximate the sensitivity using the formally derived sensitivity equation (1.26). We use the finite element method for the spatial discretization with cubic B-spline basis functions. Since these basis functions have continuous second derivatives, the second derivative evaluations $\phi_{xx}(\alpha)$ are well defined. To approximate $w_{xx}(\alpha)$ (which does not exist in the classical sense), we use the second derivative of the finite element solution of the steady problem. The numerical solutions of the formal sensitivity equation for 32 equally spaced nodes is compared with the exact sensitivity in Figure 6.

The numerical solution of the formal sensitivity equation is not close to the true sensitivity. Furthermore, as the mesh is refined this error does not decrease and the numerical sensitivity does not converge. This may not be surprising since this approach uses cubic polynomial basis functions to approximate a function whose second derivative is not smooth. Hence, the accuracy of the sensitivity computations and the reliability of the Euler-Bernoulli beam system simulations is brought into question by the results of this steady problem. The discontinuous coefficients in equations (1.22) and (1.23) along with the explicit parameter dependence of these discontinuities is the first indication that such numerical issues must be confronted for this problem. This steady problem shows that approximating a sensitivity using an ill-posed sensitivity equation may lead to large errors.

Remark: It is interesting to note that there is a direct connection between the formal sensitivity equation and automatic differentiation (AD). In our computations, we used the same number of finite element basis functions for the state equation and the formal sensitivity equation. In this case, the discretized sensitivity equation can be derived by implicitily differentiating the discretized state equation with respect to the parameter α. This is exactly how one uses AD to numerically approximate the sensitivity. Therefore, the widely used AD procedure also fails to give accurate sensitivity approximations for this problem.

In future work, the PI plans to investigate how to correctly formulate the sensitivity equation in order to obtain numerically reliable sensitivities. See Section **??** for more details.
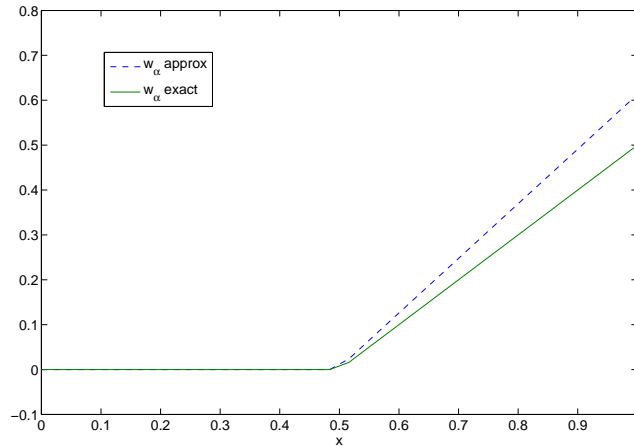
Figure 6: Finite element solution using cubic B-splines (with 32 equally spaced nodes) of the formal sensitivity equation (1.26) compared with the exact sensitivity (1.25).

1.8 Conclusions and Future Work

The heat equation example in Section 1.6 shows that sensitivity information has potential to provide valuable information for optimizing sensor and actuator location. However, the steady beam equation example in Section 1.7.1 shows very clearly that when using the continuous sensitivity equation method to compute sensitivities, the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. One goal of this research effort is to advance mathematical and computational tools to accurately compute parameter sensitivities for systems governed by partial differential equations.

For the steady equation in (1.23), and similarly for the beam example in (1.13)-(1.17), the discontinuous coefficients as well as the explicit parameter dependence of these discontinuities provide the researcher with an upfront indication that there may be a loss of regularity when moving from the original state equation to the derivation of the corresponding sensitivity equation. This is noted specifically in equation (1.26). Although this phenomenon occurs in a simple model problem, this issue is expected to appear in more realistic problems, such as the placement of actuators on a flexible wing. When sensor/actuator placement is parameterized in such a manner, the formulation and analysis of the corresponding sensitivity equations must be done carefully to insure that any regularity issues are accounted for. As discussed in Section 1.7.1, the regularity issue is of particular importance for accuracy and convergence of numerical sensitivity approximations.

For the beam problem with the discontinuous coefficients, research indicates that there are many existing mathematical options that can be used to help address the formulation of the sensitivity equation as well as the numerical computations. In particular, one can use a very weak formulation of the sensitivity equation to approximate the sensitivity variables that may lack the same smoothness as their original state variable counterparts. This approach has been successfully implemented in [23] to approximate sensitivities with respect to interface location in a heat conduction problem. Using this technique avoids the regularity issues encountered in the formally defined sensitivity equation presented in equation (1.26).

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Another potential method to treat the regularity issue of the sensitivity equations involves smoothing the discontinuous coefficients in the original mathematical model prior to the sensitivity equation derivation and applying any discretization technique. However, preliminary research using such a technique indicates that there are computational issues involved in accurately approximating the state gradients, and the smoothness of the sensitivity is still in question. This is the subject of a paper which is currently in preparation.

In addition, one can investigate the use of the immersed finite element method [57] in order to obtain numerically reliable sensitivities. Other approaches such as discontinuous Galerkin finite elements and Nitche's method are also possible. In any case, when sensor/actuator location is in question, one has to treat the derivation of the CSEM and corresponding sensitivity calculations using techniques which account for loss of smoothness in such sensitivity variables.

One other note is that if the sensitivity calculations were to be carried out using the very basic finite difference techniques, such as those used in Section 3.1.2, the lack of smoothness of the sensitivity variable would impact the accuracy and convergence rates of those computations. In addition, other sensitivity techniques such as the popular and well-advanced Automatic Differentiation (AD) software package also suffers from similar problems. Regardless of which of the popular techniques one uses for the sensitivity analysis, the mathematics of the sensor/actuator location problems must be handled carefully.

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2 Publications

Published in Peer-Reviewed Journals and Peer-Reviewed Proceedings

1. "A Petrov-Galerkin Finite Element Method for Interface Problems Arising in Sensitivity Computations," J. Burns, T. Lin and L. G. Stanley, Computers and Mathematics with Applications, Vol. 49 (2005), pages 1889-1903.

2. "Sensitivity Analysis for Actuator Placement on an Euler-Bernoulli Beam," L. G. Stanley, 43rd IEEE Conference of Decision and Control Proceedings,(Proceedings of the 43rd IEEE Conference on Decision and Control, December 14-17, 2004, Nassau, The Bahamas.) IEEE Catalog Number 04CH37601C, CD Rom ISBN:0-7803-8683-3, Paper ISBN: 0-7803-8682-5/04, Paper Reference: 0260-WeA04.6,

3. "Sensitivity Computations of Eddy Viscosity Models with an Application in Drag Computation," F. Pahlevani, International Journal for Numerical Methods in Fluids, February 2006, Vol. 52, pages 381-392.

Published in Un-Reviewed Conference Proceedings

1. "Modeling of Bioinspired Sensors for Flow Separation Detection in Micro Air Vehicles," B.T. Dickinson, J.R. Singler and B.A.Batten, in Proceedings of the 3rd AIAA Flow Control Conference, 2006, AIAA paper number 2006-3019.

Accepted for Publication

1. "Differentiability With Respect to Parameters of Weak Solutions of Linear Parabolic Equations,"J.R. Singler, *to appear in Mathematical and Computer Modelling.*

Submitted for Publication

1. "Estimation Methods for Resonance Electron Capture and Thermal Detachment Rates Using Pulsed High Pressure Mass Spectrometry," L. G. Davis, M. Larkin, M. Pernarowski, submitted to the Journal of Chemical Physics, *in revision.*

2. "Parameter Sensitivity Analysis of Eddy Viscosity Models for Navier-Stokes Equations", F. Pahlevani, *submitted to SIAM Journal on Numerical Analysis, in revision.*

3. "Semi-Implicit Schemes for Transient Navier-Stokes Equations and Eddy Viscosity Models," F. Pahlevani and L.G. Davis, *submitted to Numerical Methods for Partial Differential Equations, in revision.*

4. "Transition to Turbulence, Small Disturbances, and Sensitivity Analysis I: A Motivating Problem," J. R. Singler, *submitted to Journal of Mathematical Analysis and Applications.*

5. "Transition to Turbulence, Small Disturbances, and Sensitivity Analysis II: The Navier-Stokes Equations," J. R. Singler, *submitted to Journal of Mathematical Analysis and Applications.*

3   Technology Assists, Transitions or Transfers

None

4   Accomplishments and Successes

The research has shown that sensitivity information has the potential to provide valuable information for optimizing sensor and actuator location for control design. However, investigations into certain types of model problems also has shown that when using the continuous sensitivity equation method to compute sensitivities, the formulation and regularity of the mathematical model governing the underlying system behavior must be well understood. Various mathematical models have been investigated with varying degrees of success. Structural models such as the outlined in the report include the Euler-Bernoulli beam model with discontinuous coefficients representing actuator placement along with some models that arose due to simplification of this model. Dr. Faranak Pahlevani investigated various aspects sensitivity analysis with respect to the eddy viscosity parameter in Eddy Viscosity Models which are a subclass of Large-Eddy Simulation models for computational fluid dynamics applications. Dr. John Singler has developed a theoretical framework for the derivation of continuous sensitivity equations for a certain class of parabolic PDE models. In addition, he has collaborated with the PI on various aspects of sensitivity computation for the Euler-Bernoulli beam model. Dr. Singler has also investigated the use sensitivity analysis to explain and predict transition to turbulence for Burgers' Equation models with the goal of future application to the study of transition to turbulence for Navier-Stokes equations.

   The successes of this research effort include the advance of the mathematical and computational tools for accurate computation of parameter sensitivities for systems governed by elliptic, parabolic as welll as nonlinear elliptic partial differential equations. Through the use of such available methods as the Petrov-Galerkin Finite Element Method, accurate sensitivity computation for elliptic interface problems were shown to be a viable option. Other types of methods including Discontiuous Galerkin as well as Immersed Interface methods are also very likely to prove useful for future efforts in this area. Furthermore, the research has clearly shown that relying on black-box solvers for accurate, as well as numerically convergent, sensitivity calculations may lead to problems. The research has clearly shown that for mathematical models where the parameter of interest corresponds to locations of sensors or actuators, the derivation and simulation of sensitivity equations must be handled with caution. The main successes of the project include the discovery of examples for which standard continuous sensitivity equation methods work well and may be done somewhat automatically as well as the identification of certain types of mathematical models where the casual sensitivity equation derivation will likely lead to misleading sensitivity information. This may ultimately lead to inaccurate sensitivity information used for the placement of sensors and actuators in control systems which may ultimately lead to poor performance of various control systems. Further investigations into several of these ideas is warranted, and the PI and collaborators continue to investigate many of the ideas uncovered during the life of the funding.

5   Professional Personnel Supported

All of the following personnel were supported in a full-time capacity.

Bowman, Michael	**Graduate Student**, Montana State University
Davis, Lisa G.	**Associate Professor**, Montana State University
Larkin, Michael	**Graduate Student**, Montana State University
Pahlevani, Faranak	**Postdoc**, Montana State University
Singler, John	**Postdoc**, Montana State University/Oregon State University
Thorenson, Jennifer	**Graduate Student**, Montana State University

6   Honors and Awards

None

7   Professional Activities

The following is a list of the professional societies for which each of the primary contributors of this research is currently a member. In addition, each of these primary contributors has given invited talks at professional society meetings over the course of the grant.

Dr. Lisa G. Davis

- Society for Industrial and Applied Mathematics (SIAM)

Dr. Faranak Pahlevani

- American Mathematical Society (AMS)

- Society for Industrial and Applied Mathematics (SIAM)

Dr. John Singler

- American Institute for Aeronautics and Astronautics (AIAA)

- Society for Industrial and Applied Mathematics (SIAM)